THE PROBLEM OF AVERAGING RANDOM STRUCTURES IN TERMS OF DISTRIBUTION FUNCTIONS*

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A new approach to solving averaging problems for micro-inhomogeneous
continua, based on a restatement of the problem in terms of distribution
functions, is described. Problems having a variational structure are
considered. It is shown that, in terms of distribution functions, they
reduce to the problem of minimizing a linear functional, having the
meaning of the expectation value of the energy, in a set of distribution
functions which is distinguished by an infinite number of linear
constants. These constraints express certain matching conditions and
contain multipoint distribution functions of the random characteristics
of the medium. The constraints form an unlinked chain, the break of
which at the n-th step contains only n-point distribution functions.
In view of this, a sequence of approximate problems arises.

1. Formulation of the problem.

While there are several ways of studying composite materials, they all leave to one side
the question of the distribution functions of the microfields, say the distribution functions
of the stresses in the polycrystals. The microfields in essence remain unknown since, to
know a random field means to know the family of its distribution functions. In view of this,
it becomes necessary to state the problem of the behaviour of the material in such a way that
it is a problem of finding the family of distribution functions which characterize the material
by its microfields. As an example of a physical theory which is constructed in these terms,
the theory of rarefied gases may be mentioned, in which, as in theory of composite materials,
there are equations for the mean characteristics (the equations of gas dynamics), while to
study the microfields we have to turn to the equation for the one-point distribution function
(Boltzmann's equation). Though we can hardly count on such a simple situation in the theory
of materials, there must undoubtedly be some analogies with Boltzmann's theory. Below, we
state exactly the problem of the behaviour of composite materials in terms of distribution
functions. A sequence of problems then arises which recalls, if we continue the association
with the theory of gases, the chain of Bogolyubov-Born-Green-Kirkwood-Ivon equations.

We shall start from a statement of the material behaviour problems in terms of the
realization of random fields.

A clear mathematical statement of such problems is obtained after expressing, as in /1/
for media with a periodic structure, the idea of the asymptotic nature of the averaging
problem (see also /2, 3/ and the references cited there). The ratio of the scale of the in-
homogeneity (the cell step in the periodic case) to the characteristic scale of the problem
is a small parameter. The required functions are regarded as functions of fast and slow
variables. The fast variables vary in a cell, while the slow variables "number" the cells.
To a first approximation, to find the dependence on the fast variables, we have the so-called
problem in a cell, while the dependence on the slow variables is found from the averaged
equations. The coefficients of the averaged equations (and the form of these equations in
the case of non-linear problems) are found from the solution of the cell problem. Thus
solution of the averaging problem reduces to solution of the cell problem.

The asymptotic approach has been extended to the case of almost periodic and random media
/4, 5/. The analogue of the cell problem then proved to be the problem of the behaviour of
a continuous medium in an infinite space for a typical realization of the random character-
istics of the medium. Its statement proved to be a very important step in understanding the
averaging problem and enabled the well-known heuristic relations to be established and some
new general facts to be proved /4-10/. We shall follow /7, 8/ when stating this problem.

We consider a continuous medium defined by the Lagrangian Λ, which depends on the physical
characteristic a of the medium and on the derivatives u_i of the required function u with
respect to the space coordinates: $\Lambda = \Lambda(a, u_i)$. In the context of applications, the space is
assumed to be three-dimensional, and the subscript i covers the values 1, 2, 3; but the
dimensionality is not important in what follows. If there are several required functions and

characteristics of the medium, then u and a denote the respective sets of quantities. The Lagrangian Λ can signify the internal energy of an elastic body (a is the modulus of elasticity, and u are the components of the displacement vector), the dissipation in a heat-conducting body (a is the thermal conductivity, and u is the temperature), or the dissipation in Stokes' flow of a viscous fluid (a is the coefficient of viscosity, and u are the velocity components) etc.

We introduce the auxiliary space R of fast variables y_i, the set of which is denoted by y. We assume that the system of coordinates y_i in R is Cartesian. Given any function $\varphi(y)$ we define its mean over the space by

$$\langle \varphi(y) \rangle = \lim_{\lambda \to \infty} \frac{1}{\lambda^3} \int\!\!\!\int\limits_{-\lambda/2}^{\lambda/2}\!\!\!\int \varphi(y)\, d^3y$$

It is understood that the limits and integrals encountered throughout exist.

Every sample of the continuous medium is given by a field of characteristics $a(y)$ (see /4, 5, 7/ for details). Given any function $\psi(y)$ and numerical parameters v_i, we can define a number $\langle \Lambda(a(y), v_i + \psi_{|i}(y)) \rangle$, where the vertical bar in the subscripts denotes differentiation with respect to y_i : $\psi_{|i} = \partial\psi/\partial y_i$. The Lagrangian is lower-bounded (it can be assumed without loss of generality that $\Lambda \geqslant 0$), so that the variational problem $\inf_\psi < \Lambda(a(y), v_i + \psi_{|i})>$ is meaningful. It is non-trivial if we impose on ψ constraints, excluding the case when the $\psi_{|i}$ are identically constant, or otherwise, e.g., for strictly convex non-negative functions Λ, $\Lambda(a(y), 0) = 0$, the minimum is reached at the point $v_i + \psi_{|i} = 0$. In particular, it can be assumed that $\psi(y)$ is bounded at infinity. The problem then arising is a cell problem for a random structure. The macro-Lagrangian $\overline{\Lambda}$ depends on the macrovariables v_i, which signify the derivatives of the macrofields v, and is given by /7, 11, 4, 5/

$$\overline{\Lambda}(v_i) = \inf_{\psi(y)} \langle \Lambda(a(y), v_i + \omega_{|i}(y)) \rangle \tag{1.1}$$

where the infimum is sought in the set of functions $\psi(y)$ bounded at infinity. The v_i in (1.1) are regarded as parameters. If $a(y)$ is a periodic function, it can be shown /7/ that the minimum in (1.1) need only be sought among periodic functions $\psi(y)$ and (1.1) becomes the expression obtained in /11/ for periodic structures: the angle brackets are then understood as signifying the average over a cell.

A simple example is the problem for a heat-conducting body at rest. Here, $\Lambda(a, u_i) = \frac{1}{2}a^{ij}u_iu_j$ (summation is carried out over repeated subscripts or superscripts), a^{ij} are the thermal conductivities, while, since the problem is linear, the macro-Lagrangian $\overline{\Lambda}$ is quadratic in the gradients of the mean temperature v_i: $\overline{\Lambda} = \frac{1}{2}\bar{a}^{ij}v_iv_j$, $\bar{a}^{ij}$ are the effective thermal conductivities, which are found from the variational problem

$$\frac{1}{2}\bar{a}^{ij}v_iv_j = \inf_{\psi(y)} \langle \frac{1}{2}a^{ij}(v_i + \psi_{|i})(v_j + \psi_{|j}) \rangle \tag{1.2}$$

By choosing different samples of the random field $a(y)$ we can in general obtain different values of the macro-Lagrangian $\overline{\Lambda}$. For problem (1.2), the following important assertion was proved in /5/: if $a(y)$ is a homogeneous ergodic random field, then the random fields $\psi_{|i}$, realizing the minimum in (1.2), are also homogeneous and ergodic, while $\overline{\Lambda}$ is independent of the choice of sample of the field $a(y)$. We assume henceforth that this assertion holds for all the problems considered.

Since $\psi(y)$ is bounded at infinity, we have /7/

$$\langle \psi_{|i}(y) \rangle = 0 \tag{1.3}$$

For homogeneous ergodic fields it can happen that the variational problem (1.1), in which the condition that $\psi(y)$ be bounded at infinity is replaced by condition (1.3), has the same solution as the initial problem. Later, for clarity, we consider problem (1.1) under condition (1.3), which is easier to take into account, and it will become clear how to replace this by the condition that $\psi(y)$ be bounded.

The aim of our subsequent constructions is to restate problem (1.1) in terms of distribution functions.

2. Some heuristic considerations.

We introduce the density $f(a, \psi_i)$ of the joint distribution of fields $a(y)$ and $\psi_i(y)$ (we shall retain a bar in the subscript when we wish to emphasize specially that ψ_i is a potential vector field). Since the random fields a, ψ_i are homogeneous, the function f is independent of u, and since they are ergodic, the minimized functional can be written as

$$\langle \Lambda (a (y), v_i + \psi_i (y)) \rangle = \int \Lambda (a, v_i + \psi_i) f (a, \psi_i) \, da \, dv_\psi \tag{2.1}$$

where da and dv_ψ are elements of integration in the spaces of variables a and ψ_i.

Condition (1.3) is rewritten in terms of distribution functions as

$$\int \psi_i f (a, \psi_i) \, da \, dv_\psi = 0 \tag{2.2}$$

The function $f (a, \psi_i)$ also satisfies the natural conditions

$$\int f (a, \psi_i) \, dv_\psi = f_0 (a), \quad f (a, \psi_i) \geqslant 0 \tag{2.3}$$

where $f_0 (a)$ is the given one-point distribution function of the homogeneous field $a (y)$.

Note that (2.1) is a linear functional, call it $I (f)$, of the distribution function. Relations (2.2) and (2.3) form a system of linear constraints on f. It is obvious that the search for the minimum over the fields $\psi (y)$ is equivalent to minimizing $I (f)$ with respect to the distribution functions of this field, and we can write

$$\bar{\Lambda} (v_i) = \inf_f I (f) \tag{2.4}$$

We shall seek the minimum in (2.4) with respect to all functions f which satisfy constraints (2.2), (2.3). The problem can be solved explicitly. For simplicity, we consider the case of an isotropic $(a^{ij} = a \delta^{ij})$ heat-conducting medium. Introducing the Lagrange multiplier λ^i for condition (2.2), we arrive at the variational problem

$$\inf_f \int [^1/_2 a (v_i + \psi_i) (v^i + \psi^i) - \lambda^i \psi_i] f (a, \psi_i) \, da \, dv_\psi \tag{2.5}$$

where the minimum is sought with respect to all functions f which satisfy conditions (2.3).

The structure of problem (2.5) is as follows: it is required to find the minimum of the functional

$$\int \Lambda (t, x) f (t, x) \, dt \, dx$$

with respect to non-negative functions f with a given value for each t of the integral of f with respect to x; denote the integral by $f_0 (t)$. It is clear how the minimizing function $f^\vee$ is constructed: for each t we have to find the point $x^\vee (t)$ at which the function $\Lambda (t, x)$ has its minimum value with respect to x, and concentrate $f^\vee (t, x)$ at this point: $f' (t, x) = f_0 (t) \delta (x - x^\vee (t))$ ($\delta (x)$ is the Dirac delta function).

The solution of problem (2.5) thus reduces to minimizing the expression in the square brackets in (2.5) with respect to ψ_i. The minimizing element $\psi_i^\vee$ is given by the equation $a (v_i + \psi_i^\vee) = \lambda_i$, so that $f^\vee (a, \psi_i) = f_0 (a) \delta (\psi_i - a^{-1} \lambda_i + v_i)$. The Lagrange multiplier λ_i is found from condition (2.2); $\lambda_i = (Ma^{-1})^{-1} v_i$ (M denotes the expectation value). Substituting $f^\vee$ into (2.4), we find the minimum value of the functional; $^1/_2 (Ma^{-1})^{-1} v_i v^i$. We have thus arrived at Rice's expression, which gives a lower bound for the effective heat conduction, though it is not exactly true for any macro-isotropic structure (except for the trivial case $a (y) \equiv$ const). It can be assumed that this is true for two reasons; first, instead of using all the information about the field $a (y)$, we only used its one-point distribution function; and second, we did not take into account that ψ_i are the derivatives of a function ψ, which in general can lead to extra constraints on function $f (a, \psi_i)$.

We first consider the statement of these extra constraints, which is of independent interest.

3. Constraints on the distribution function.

Let $f_n = f (y^{(1)}, \psi^{(1)}, \psi_i^{(1)}; \ldots; y^{(n)}, \psi^{(n)}, \psi_i^{(n)})$ be the n-point distribution function of random fields ψ, ψ_i. The family of distribution functions $(n = 1, 2, \ldots)$ satisfies the following matching conditions: for all n

$$\text{the function } f_n \text{ is symmetric with respect to any pair of arguments, } f_n \geqslant 0 \tag{3.1}$$

$$f_{n-1} = \int f_n \, d\psi^{(n)} \, dv_{\psi^{(n)}}, \quad \int f_1 \, d\psi \, dv_\psi = 1 \tag{3.2}$$

The arguments f_n in (3.1) are understood to be the quantities distinguished in the writing of f_n by a point with a comma. We have the following theorem.

Theorem 1. In order for the family of distribution functions of the fields ψ, ψ_i to be a a family of distribution functions of the random field $\psi (y)$ and its derivatives $\psi |_i (y)$, it is necessary that the f_n satisfy the matching conditions (3.1) and (3.2), while for $f_2 = f (y, \psi, \psi_i; y', \psi', \psi_i')$ we must have the equation

$$\frac{\partial^2}{\partial y^i \partial y'^j} \int f_2 \, dv_\psi \, dv_{\psi'} + \frac{\partial}{\partial y'^j} \frac{\partial}{\partial \psi} \int f_2 \psi_i \, dv_\psi \, dv_{\psi'} +$$
$$\frac{\partial}{\partial y^i} \frac{\partial}{\partial \psi'} \int f_2 \psi_j' \, dv_\psi \, dv_{\psi'} + \frac{\partial^2}{\partial \psi \partial \psi'} \int f_2 \psi_i \psi_j' \, dv_\psi \, dv_{\psi'} = 0 \qquad (3.3)$$

Here, $dv_{\psi'}$ is a volume element in the space of variables ψ_i'. It is assumed that all the integrals and derivatives in (3.3) exist. If, in addition, $f_2 \to 0$ as $\psi \to \infty$, $\psi' \to \infty$ so fast that the following integrals over the sphere Σ_r or radius r in the space of variables ψ ($d\sigma_\psi$ is a volume element of Σ_r) tend to zero as $r \to \infty$:

$$\int_{\Sigma_r} \psi \psi_i \psi' f_2 \, d\sigma_\psi \, d\psi' \, dv_\psi \, dv_{\psi'}, \quad \int_{\Sigma_r} \psi \psi_i \psi_j f_2 \, d\sigma_\psi \, dv_\psi \, dv_{\psi'}$$

then the conditions mentioned are also sufficient.

Notes. 1°. We write (3.3) under the assumption that f_2 is differentiable. Otherwise, (3.3) must be understood in the weak sense, i.e., as an integral identity.

2°. If ψ is a vector, and not a scalar, field, then $\partial/\partial\psi$ in (3.3) must be understood as a divergence, e.g., the form (summation over α, β)

$$\frac{\partial^2}{\partial \psi^\alpha \partial \psi'^\beta} \int \psi_i^\alpha \psi_j'^\beta f_2 \, dv_\psi \, dv_{\psi'}$$

Proof. Necessity. Let $\alpha(y, \psi; y', \psi')$ be a finite function of its arguments, and $\psi(y)$ a realization of a differentiable random field ψ. We construct the function $\beta(y, y')$ of two arguments: $\beta(y, y') \equiv \alpha(y, \psi(y); y', \psi(y'))$. Since α is finite in y, y', then $\beta(y, y')$ is also finite in y, y', so that

$$\int \frac{\partial^2 \beta(y, y')}{\partial y^i \partial y'^j} \, d^3y \, d^3y' = 0$$

Writing this equation in terms of function α, we have

$$\int \left(\frac{\partial^2 \alpha}{\partial y^i \partial y'^j} + \frac{\partial^2 \alpha}{\partial \psi \partial y'^j} \frac{\partial \psi(y)}{\partial y^i} + \frac{\partial^2 \alpha}{\partial \psi' \partial y^i} \frac{\partial \psi(y')}{\partial y'^j} + \right.$$
$$\left. \frac{\partial^2 \alpha}{\partial \psi \partial \psi'} \frac{\partial \psi(y)}{\partial y^i} \frac{\partial \psi(y')}{\partial y'^j} \right) d^3y \, d^3y' = 0 \qquad (3.4)$$

Taking the expectation value of the left-hand side, we have

$$\int \left(\frac{\partial^2 \alpha}{\partial y^i \partial y'^j} + \frac{\partial^2 \alpha}{\partial \psi \partial y'^j} \psi_i + \frac{\partial^2 \alpha}{\partial \psi' \partial y^i} \psi_j' + \right.$$
$$\left. \frac{\partial^2 \alpha}{\partial \psi \partial \psi'} \psi_i \psi_j' \right) f_2 \, d\psi \, d\psi' \, dv_\psi \, dv_{\psi'} \, d^3y \, d^3y' = 0 \qquad (3.5)$$

Since $\alpha(y, \psi; y', \psi')$ is arbitrary, (3.3) follows from (3.5). The necessity of matching conditions (3.1), (3.2) is well-known, see e.g., /12/.

Sufficiency. Assume that we have a family of distribution functions that satisfy the matching conditions (3.1) and (3.2). Then, by Kolmogorov's theorem /12/, there are random fields $\psi(y, \omega)$, $\psi_i(y, \omega)$ (ω is an element of probability space) whose distribution functions belong to this family. We show that, with probability one, by virtue of Eq. (3.3), the functions $\psi_i(y, \omega)$ are the derivatives with respect to y^i of $\psi(y, \omega)$. For this, it suffices to show that

$$M \left[\psi(y, \omega) - \psi(y', \omega) - \int_\Gamma \psi_i(y, \omega) \, dy^i \right]^2 = 0 \qquad (3.6)$$

where Γ is a contour joining the points y and y'. Removing the brackets in (3.6) and taking the expectation value, we obtain

$$B(y, y) + B(y', y') - 2B(y, y') + \int_\Gamma \int_\Gamma B_{ij}(z, z') \, dz^i \, dz'^j - \qquad (3.7)$$
$$2 \int_\Gamma \int_\Gamma \psi_i' \psi (f(y, \psi, \psi_i; z, \psi', \psi_i') - f(y', \psi, \psi_i; z, \psi', \psi_i')) \times$$
$$d\psi \, d\psi' \, dv_\psi \, dv_{\psi'} \, dz^i = 0, \quad B(y, y') = M\psi(y, \omega) \psi(y', \omega),$$
$$B_{ij}(y, y') = M\psi_i(y, \omega) \psi_j(y', \omega)$$

We now multiply (3.3) by $\psi\psi'$ and integrate the result with respect to $\psi\psi'$, and then integrate over the contour Γ with respect to y^i and y'^j. It can be shown that Eq.(3.7) is then obtained (we need the condition for the integrals over Σ_r to tend to zero, in order for the terms that appear in the integration by parts to vanish). Thus, (3.6) follows from (3.3), and for (almost) every sample

$$\psi(y, \omega) - \psi(y', \omega) - \int_\Gamma \psi_i(z, \omega)\, dz^i = 0$$

Consequently, $\psi_i(y, \omega)$ are the derivatives with respect to y^i of the function $\psi(y, \omega)$, which it was required to prove.

Theorem 2. If ψ_i is the gradient of the field ψ, we have for the one-point distribution function $f(y, \psi, \psi_i)$

$$\frac{\partial}{\partial y^i} \int f\, dv_\psi + \frac{\partial}{\partial \psi} \int \psi_i f\, dv_\psi = 0 \tag{3.8}$$

Eq.(3.8) holds in the same way as (3.3).

We now consider the homogeneous random fields for which f_1 is independent of y, while f_2 depends on the coordinates only via the difference $\tau = y' - y$. We assume that $f_2 \to 0$ so fast as $\psi \to \infty, \psi' \to \infty$, that, when integrating by parts expressions of the type $\psi \partial f_2/\partial \psi$, the term outside the integral vanishes.

Corollaries. 1°. For homogeneous random fields

$$\frac{\partial}{\partial \psi^\alpha} \int f(\psi^\alpha, \psi_i^\alpha)\, \psi_i^\alpha\, dv_\psi = 0 \tag{3.9}$$

The superscript α, which numbers the set of functions ψ_i is restored in (3.9) if there are several of them, in order to emphasize that Eq.(3.9) signifies the "incompressibility" for every i of the mean field of gradients in the space of variables ψ^α.

2°. The mean value of the field gradient is zero:

$$\int f(\psi, \psi_i)\, \psi_i\, d\psi\, dv_\psi = 0 \tag{3.10}$$

We obtain (3.10) from (3.9) by multiplying by ψ^α and integrating by parts.

3°. For homogeneous fields f_2 we have the equation

$$-\frac{\partial^2}{\partial \tau^i \partial \tau^j} \int f_2\, dv_\psi\, dv_{\psi'} + \frac{\partial}{\partial \tau^j}\frac{\partial}{\partial \psi} \int f_2 \psi_i\, dv_\psi\, dv_{\psi'} - \tag{3.11}$$
$$\frac{\partial}{\partial \tau^i}\frac{\partial}{\partial \psi'} \int f_2 \psi_j'\, dv_\psi\, dv_{\psi'} + \frac{\partial^2}{\partial \psi \partial \psi'} \int f_2 \psi_i \psi_j'\, dv_\psi\, dv_{\psi'} = 0$$

4°. We have the equation

$$\sum_\alpha \int f_2(\psi_i; \tau, \psi_i')\, \psi_i^\alpha \psi_j'^\alpha dv_\psi\, dv_{\psi'} = \frac{\partial^2 B(\tau)}{\partial \tau^i \partial \tau^j} \tag{3.12}$$
$$B(\tau) = \sum_\alpha M\psi^\alpha(y)\, \psi^\alpha(y + \tau)$$

It is obtained from (3.11) by multiplying by ψ, ψ' and integrating with respect to ψ, ψ'.

Theorem 3. The necessary and sufficient conditions for the random homogeneous field $\psi_i(y, \omega)$ to be a field of derivatives of a random function $\psi(y, \omega)$ are that the family of distribution functions satisfies matching and homogeneity conditions and that there is a function $B(\tau)$ such that (3.12) holds.

The proof is similar to that of Theorem 1, the only difference being that, instead of (3.6), we establish the equation (Γ is any closed contour)

$$\sum_\alpha M\left(\oint_\Gamma \psi_i^\alpha(z, \omega)\, dz^i\right)^2 = 0$$

4. The averaging problem in terms of distribution functions.

In Sect.3 we obtained all the relations needed for restating the problem in a cell. Let $f_n(y^{(1)}, a^{(1)}, \psi_i^{(1)}; \ldots; y^{(n)}, a^{(n)}, \psi_i^{(n)})$ $(n = 1, 2, \ldots)$ be the family of n-point distribution functions of random fields $a(y), \psi_i(y)$, which satisfy the matching conditions. The field $a(y)$ is given, so that the functions f_0 in the equation

$$\int f\left(y^{(1)}, a^{(1)}, \psi_i^{(1)}; \ldots ; y^{(n)}, a^{(n)}, \psi_i^{(n)}\right) dv_{\psi^{(1)}} \ldots dv_{\psi^{(n)}} =$$
$$f_0\left(y^{(1)}, a^{(1)}; \ldots ; y^{(n)}, a^{(n)}\right) \tag{4.1}$$

are known.

We consider the problem of minimizing the functional $I(f)$ in the set of families of distribution functions which is distinguished by the constraints (2.2), (3.12), and (4.1), and we assume that the lower bound of $I(f)$ is reached in this set.

Theorem 4. Eq.(2.4) holds, in which inf is sought over all the distribution functions that satisfy the matching conditions and conditions (2.2), (3.12), and (4.1).

Proof. Let $\psi^\smile(y)$ be the minimizing element of the cell problem. Let f^* denote the family of distribution functions of the fields $a(y)$, $\partial\psi^\smile(y)/\partial y^i$. This family satisfies the matching conditions and (2.2), (3.12), (4.1), so that

$$\inf_f I(f) \leqslant I(f^*) = \overline{\Lambda}(v_i) \tag{4.2}$$

Now let $f^\smile$ be the minimizing family of distribution functions in our problem. Since it satisfies the matching conditions, there exist by Kolmogorov's theorem random fields $\psi_i(y, \omega)$ for which these functions are distribution functions. Since Eq.(3.12) holds, there is a random field $\psi(y, \omega)$ for which $\psi_i = \partial\psi(y, \omega)/\partial y^i$. Moreover, $M\Lambda(a, v_i + \psi_i) = \langle\Lambda(a, v_i + \psi_i)\rangle < \infty$. On taking $\psi(y, \omega)$ as the test function of the cell problem, we obtain

$$\overline{\Lambda}(v_i) \leqslant \langle\Lambda(a, v_i + \psi_i)\rangle = M\Lambda(a, v_i + \psi_i) = \inf_f I(f). \tag{4.3}$$

From inequalities (4.2) and (4.3) we obtain the theorem.

Notes. 3°. Our variational problem is a problem of minimizing a functional $I(f)$ which is linear in f, in a set of functions which satisfy linear constraints, i.e., in this case it is a linear programming problem. Note that our theorem reduces both linear and non-linear (with non-quadratic Lagrangians) problems to a linear programming problem.

4°. Without dwelling on the fact in detail, note that the cell problem for a periodic structure also reduces to a linear programming problem. Here, by the probability of the values of field ψ_i lying in a domain A of the space of variables ψ_i, we have to understand the volume part of the cell in which ψ_i takes values from A. Relation (2.1) then refers to finding the Lebesgue integral over the cell. The interpretation of n-point distribution functions is similar.

5. The sequence of approximations.

The above statement of the averaging problem contains an infinite number of constraints. In this connection a sequence of approximation problems arises, in each of which there is only a finite number of constraints. In the problem of the first appproximation, only one-point distribution functions participate, in the second, only two-point functions, etc. In each successive problem, more and more detailed characteristics of the random field $a(y, \omega)$ are taken into account.

Let $I_{(n)}$ be the lower bound of the functional $I(f)$ in the set distinguished by the constraints of all the k-point distribution functions with $k \leqslant n$.

Consider the possibility of the equation

$$\lim_{n\to\infty} I_n = \overline{\Lambda}(v_i) \tag{5.1}$$

As n increases, the set of admissible functions contracts, so that $I_{(n)} \leqslant I_{(n+1)}$. By (4.2), the sequence $I_{(n)}$ is upper bounded by $\overline{\Lambda}(v_i)$. It therefore has a limit. Assume that it is less than $\overline{\Lambda}(v_i)$. Let $f_{(n)}^\smile$ be the minimizing element of the n-th approximation problem. If there is a field $\psi^{(n)}(y)$ and a field $a^{(n)}(y)$ such that $f_{(n)}^\smile$ is the joint distribution of $a^{(n)}(y)$ and $\psi_i^{(n)}(y)$, then $I_{(n)} = \langle\Lambda(a^{(n)}(y), v_i + \psi_i^{(n)}(y))\rangle$. As $n \to \infty$, the fields $a^{(n)}(y)$ and $a(y)$ approximate in their distributions, and for sufficiently large n we have $I_{(n)} = \langle\Lambda(a(y), v_i + \psi_i^{(n)})\rangle + o(n)$. On taking $\psi^{(n)}$ as the test field in the cell problem, we obtain the bound $\overline{\Lambda}(v_i) \leqslant I_{(n)} + o(n)$, which leads to a contradiction with our assumption.

If we introduce into the space of one-point distribution functions the "energy norm" (2.1) (recall that $f \geqslant 0$, $\Lambda \geqslant 0$), then Eq.(5.1) can be interpreted as convergence in the energy norm of one-point distribution functions of the approximate problems to the one-point function of the initial problem. The question of the convergence of many-point distribution functions remains open.

The first three approximations are of greatest interest; higher order approximations are scarcely necessary.

6. The dual problem.

The approximation error can be efficiently monitored by using the dual problem. In terms of realizations, it can be stated as

$$\overline{\Lambda}(v_i) = \sup_{p^i} (\langle p^i \rangle v_i - \langle \Lambda^*(a(y), p^i(y)) \rangle) \tag{6.1}$$

where the sup is sought over all constraints at infinity on the vector fields $p^i(y)$ which satisfy the constraint

$$p^i_{|i}(y) = 0 \tag{6.2}$$

and $\Lambda^*(a, p^i)$ is the Young-Fenchel transformation of $\Lambda(a, u_i)$ with respect to the variables u_i:

$$\Lambda^*(a, p^i) = \sup_{u_i}(p^i u_i - \Lambda(a, u_i))$$

The general solution of Eq.(6.2) can be written as

$$p^i = e^{ijk} \varphi_{j|k}$$

where e^{ijk} are Levi Civita symbols. The dual can therefore be written in the same form as the initial problem, with the integrand $L(a, \varphi_i, v_i) = e^{ijk} \varphi_{i|k} v_i - \Lambda^*(a(y), e^{ijk} \varphi_{j|k})$. The statement of the dual problem in terms of distribution functions is the same as for the initial problem, with Λ replaced by L, inf by sup, and ψ by φ_i.

Let $\overline{J}_{(n)}$ denote the upper bound of the functional

$$J = \int L(a, \varphi_i, v_i) f(a, \varphi_i) \, da \, dv_\varphi$$

in the set distinguished by the constraints on the k-point distribution functions, $k = 1, \ldots, n$. In the same way as in Sect.5, we can shown that

$$\lim_{n \to \infty} \overline{J}_{(n)} = \overline{\Lambda}(v_i)$$

The sequence $\overline{J}_{(n)}$ is decreasing: $\overline{J}_{(n+1)} \leqslant \overline{J}_{(n)}$, and moreover $\overline{\Lambda}_{(n)} \leqslant \overline{\Lambda}(v_i) \leqslant \overline{J}_{(n)}$. These inequalities can provide two-sided bounds on the macro-Lagrangian.

7. Approximations and extremality with respect to microstructures.

We specify an n-point distribution function of the field $a(y)$ and denote by A_n the set of fields $a(y)$ which have this distribution function. Since the macro-Lagrangian $\overline{\Lambda}(v_i)$ depends on the choice of field $a(y)$, we note this by assigning to it the subscript a: $\overline{\Lambda} = \overline{\Lambda}_a(v_i)$. We state without proof the following property of n-th approximation problems, which explains their significance:

$$\underline{I}_{(n)} = \inf_{a \in A_n} \overline{\Lambda}_a(v_i), \quad \overline{J}_{(n)} = \sup_{a \in A_n} \overline{\Lambda}(v_i)$$

It can be shown that all the well-known results of averaging theory, which contain one-point distribution functions (Dykhne's formula, the Lur'e-Cherkayev estimates, etc.), can be extracted from the first approximation problem. The second approximation cannot be studied analytically; it seems only to be open to numerical methods.

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ON THE STRUCTURE OF QUASITRANSVERSE ELASTIC SHOCK WAVES[*]

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The structure of quasitransverse shock waves in a slightly anisotropic
medium in the presence of dissipation due to viscosity is investigated.
The existence of a shock structure "responsible" for ambiguity of the
solution of a selfsimilar problem about waves excited in a half-space is
demonstrated. The question of the existence of a structure for the
remaining quasitransverse shock waves is discussed.

It is shown in the analysis of gas dynamics /1/ and certain other /2-5/ problems that
selection of the discontinuities that should be utilized to construct solutions should not be
constrained, in many cases, by just conditions for a non-decrease in entropy and by evolution-
arity conditions. Confirmation of the requirement often utilized for the existence of the
structure of a discontinuity /1, 3, 4, 5/ is especially important and interesting in cases
when ambiguity of the solution of problems occurs, as in /1-4/ since it assists in selecting
the unique solution that can actually be realized.

Ambiguity of the solution for certain ranges of the problem parameters was detected /6,
7/ when constructing solutions of selfsimilar problems in a prestressed or generally weakly
anisotropic elastic medium for given initial strains and strains different from the initial
on the half-space boundary. Quasitransverse shock waves that satisfy the condition of a non-
decrease in entropy and the evolutionarity conditions were utilized in constructing the
solutions. It can be suspected that, as in /1-4/, the ambiguity of the solution of self-
similar problems is a result of the fact that not all the shock waves mentioned possess a
structure, i.e., a continuous solution of a certain more-complete system of equations taking
account of the dissipative processes proceeding in a narrow zone corresponding to the dis-
continuity in the solution of the original equations.

We will make a general remark here concerning the further content of the research where
(as in /1-4/) only a stationary shock wave structure is examined. In cases when there is no
stationary structure and the appropriate discontinuity necessarily occurs in the solution of
the problem, a non-stationary structure is apparently realized. A well-known example of this
kind is the hydraulic jump whose structure is turbulent. In those cases when there are several
solutions of the problems, preference should obviously be given to solutions containing dis-
continuities possessing a stationary structure (as is done in /1-5/). From this viewpoint,
shocks encountered in the solution only in the case of ambiguity, when tere is a competing
solution not containing a shock of this kind, are of greatest interest for investigating the
stationary structure.

The structure of quasitransverse shocks for which the necessary conditions for existence
(evolutionarity and non-decrease of the entropy) are satisfied /8, 9/, is investigated below.
Terms taking account of the additional stresses caused by viscosity are added as a dissipative
mechanism to the dynamical elasticity theory equations. This is the simplest of the dis-
sipative mechanisms used in the theory of a solid deformable body and ensures continuity of
the solutions. It can be hoped that if ambiguity of the solution of the selfsimilar problem
is associated with the absence of a structure for part of the discontinuities, then this should
already have been detected in this model.

1. We consider the motion in the form of plane waves parallel to a certain plane which
we select as the coordinate plane $x_1 x_2$ of a Lagrange coordinate system $x_1, x_2, x_3 = x$. The quan-
tities x_1, x_2, x correspond to rectangular Cartesian coordinates in the non-deformed state of
the medium. Initial deformation of the medium, if it exists, is considered homogeneous: